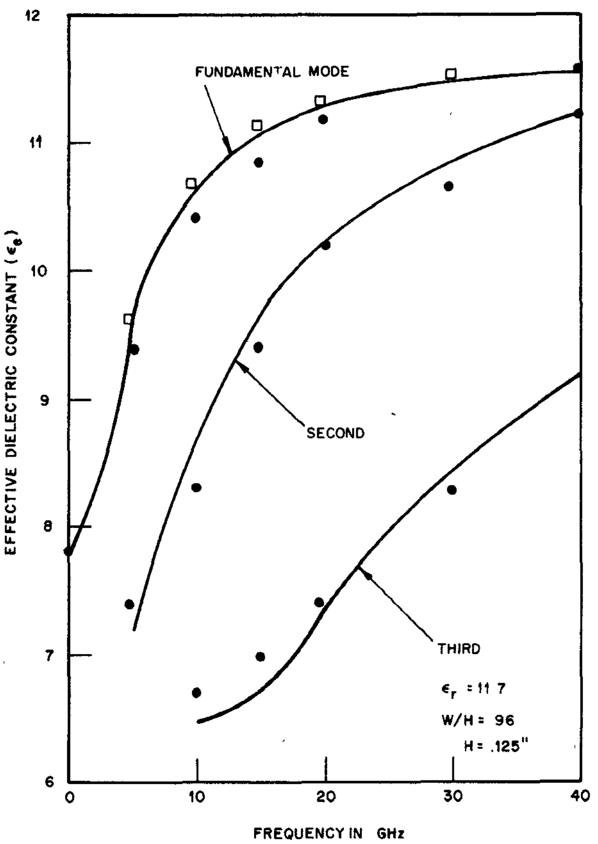

Fig. 5. Normalized propagation constant for fundamental and higher order modes in open microstrip (● Van de Capelle and Lypaert [12], □ Getsinger [14]).

V. Conclusions

A continuous-spectrum method has been used in conjunction with the method of moments to treat the time-harmonic solution of covered and uncovered microstrips. Both longitudinal and transverse currents were considered in the analysis. The propagation constants for fundamental and higher order modes in open microstrip were calculated. In each case the results are in good agreement with available theoretical and experimental data. The results are accurate to within 4 percent.

Acknowledgment

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General Noise Analysis for Bias- and RF-Voltage-Dependent Transferred-Electron Devices

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Abstract—**The general AM and FM noise spectrum analysis of Sweet for transferred-electron devices is extended to include the variation of device admittance with both bias- and RF-voltage amplitudes. This is important because recent investigations by the author suggest that there are significant variations of device admittance with both parameters. Also the expressions for the AM and FM noise spectra are formulated in terms of the more basic quantities such as stored charge, modulation sensitivities, and incremental admittance.**

Introduction

The lumped-circuit analysis of noise in self-excited oscillators has received considerable attention. Edson [1], Mullen [2], and van der Pol [3] wrote basic papers on this subject. As different self-oscillating devices have been developed, their noise properties have been studied in detail. Lax [4] underscored this individuality of self-excited oscillators when he observed that "the noise mixes with the signal in a complex fashion that is quite different from ordinary nonlinear systems .... It is not satisfactory to represent the spectrum as a delta function signal plus a background. The noise will spread the delta function spectrum into a finite width." This complex mixing is dependent both on the device properties and on the device environment. Hence, it is necessary to combine an oscillator device model with an RF-circuit model to completely study oscillator noise properties. This short paper extends the theoretical groundwork for the general noise analysis of transferred-electron (TE) devices.

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The individuality of self-excited oscillators underscores the need to supplement the general publications on nonlinear solid-state-device noise properties [5], [6]. IMPATT diodes, which are basically high-impedance devices, are best treated in equivalent-series RF circuits and have been studied extensively [7]–[13]. More recently, attention has been directed toward TE-device noise properties [14], [15], which are best studied in equivalent parallel RF circuits since TE devices are basically low-impedance oscillators. This short paper extends the parallel-circuit noise analysis of Sweet [14] to include the variation of TE-device admittance with both RF and bias voltages. Recent studies by the author show that both parameters are important.

Not only is Sweet's model an oversimplification for some important cases, but also his assumption of a capacitance to represent the device susceptance leads to an ambiguity in choosing between average and incremental capacitance when in fact the primary reactive quantity obtained from the device model chosen is susceptance.

The interpretation of the capacitance is conveniently bypassed when the noise theory is formulated using susceptance as in the following.

General Noise Analysis

General expressions are derived for the AM and FM oscillator noise of the circuit in Fig. 1 assuming that lumped RF and video or low-frequency noise sources are known. Fig. 1 includes two sources of noise. One is a narrow-band RF source $i(t)$ and the other arises from fluctuations $\Delta V_0(t)$ in the bias voltage V_B. They are assumed to be described as

$$i(t) = -i_1(t)\sin(\omega_0 t + \phi) + i_2(t)\cos(\omega_0 t + \phi) \quad (1)$$

and

$$V_B(t) = V_0 + \Delta V_0(t) \quad (2)$$

where $i_1(t)$, $i_2(t)$, and $\Delta V_0(t)$ are zero-mean random processes with known spectra, ω_0 is the oscillator frequency, ϕ is the oscillator phase, and V_0 is the dc bias voltage. Also the circuit voltage $V(t)$ is assumed to be

$$V(t) = V_{RF}(t)\cos(\omega_0 t + \phi) \quad (3)$$

where $V_{RF}(t) = V_1 + \Delta V_1(t)$ is a slowly varying amplitude with average value V_1 and fluctuation $\Delta V_1(t)$. Thus $i_1(t)$ and $i_2(t)$ are the quadrature and in-phase RF-noise components, respectively. The statistics of the random processes $\Delta V_1(t)$ and ϕ determine the AM- and FM-noise spectra and will be expressed in terms of the statistics of i_1, i_2, and ΔV_1.

The Kirchhoff current equation for the circuit voltage is

$$i(t) = -G_d V_{RF}\cos(\omega_0 t + \phi) - B V_{RF}\sin(\omega_0 t + \phi) + \left(C\frac{d}{dt} + G_0 + G_L + \frac{1}{L}\int dt\right) V_{RF}\cos(\omega_0 t + \phi). \quad (4)$$

The sign of the device susceptance term is negative because, physically, the device has a capacitive (i.e., positive) susceptance.

The excitations ΔV_1 and i are assumed small so that they may be treated as leading to perturbations about the noiseless, steady-state operating point. At this operating point the time dependence of the slowly varying terms drops out and, since the quadrature components in (4) must separately vanish, V_0 and V_1 are determined from

$$G_0 + G_L - G_d(V_1, V_0) = 0 \quad (5)$$

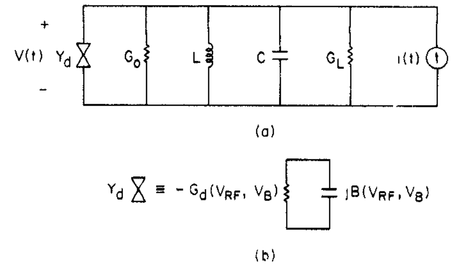

Fig. 1. Nonlinear device in a parallel circuit. (a) Parallel oscillator circuit. (b) Parallel nonlinear-device model.

and

$$\omega_0 C + B(V_1, V_0) - \frac{1}{\omega_0 L} = 0. \quad (6)$$

Perturbations about the operating point $\{V_1, V_0\}$ are expressed in the time dependence of ΔV_1 and ϕ through the sources i_1, i_2, and ΔV_0. Keeping only the first-order terms in (4) and then multiplying by the operator $(2/T)\int_t^{t+T} dt \cos(\omega_0 t + \phi)$ for $T = 2\pi/\omega_0$ gives the in-phase constraint

$$i_2(t) = -[(\partial_{V_1}G_d)\,\Delta V_1 + (\partial_{V_0}G_d)\,\Delta V_0]V_1 + \left(C + \frac{1}{\omega_0^2 L}\right)\Delta V_1' \quad (7)$$

where the prime denotes differentiation with respect to time and ∂_x denotes differentiation with respect to the variable x. Similarly, applying the operator $(2/T)\int_t^{t+T} dt \sin(\omega_0 t + \phi)$ gives the quadrature phase constraint

$$-i_1(t) = -[(\partial_{V_1}B)\,\Delta V_1 + (\partial_{V_0}B)\,\Delta V_0](V_1 + \Delta V_1) - C\phi'(V_1 + \Delta V_1) - \frac{\phi'}{\omega_0^2 L}(V_1 + \Delta V_1). \quad (8)$$

In (7) and (8) ΔV_0, i_1, and i_2 are the sources while ΔV_1 and ϕ are the responses to those sources. Inspection reveals that (7) is an independent amplitude constraint for ΔV_1 and (8) is a coupled constraint (to ΔV_1) for ϕ. This coupling is absent in Sweet's formulation and hence the expression for FM noise in the present development has an additional term depending on the in-phase noise source which Sweet's development lacks altogether, and has a slightly different video-noise term from Sweet's. The quadrature-noise terms in the FM spectrum and the AM spectrum are similar to those in Sweet's analysis. The separation of the in-phase and quadrature-phase constraints into oscillator-amplitude and oscillator-phase constraints, respectively, is familiar in van der Pol-type [3] problems. The previous inclusion of RF-voltage dependence leads to a coupling between the amplitude and phase constraints which is not present in the van der Pol analysis. That is to say, the Sweet and van der Pol analyses are isomorphic and differ from the present analysis in similar ways.

Since (7) and (8) are linear, superposition applies. The first can be solved for the spectrum $S_{\Delta V_1}(\omega_m)$ of ΔV_1. Since i_1, i_2, and ΔV_0 are assumed uncorrelated, the spectrum of ΔV_1 will be the sum of the individual contributions due to ΔV_0 and i_2. $S_{\Delta V_1}(\omega_m)/V_1^2$ is just the oscillator AM-noise spectrum $S_{AM}(\omega_m)$.

The theory of random processes applied to linear systems [16], [17] shows that for a random-process input $x(t)$ to a linear system with response $H(\omega)$, which gives the output random process $Y(t)$, the input and output spectra are related by

$$S_Y(\omega) = |H(\omega)|^2 S_x(\omega). \tag{9}$$

Using this result with the amplitude constraint (7), gives the AM-noise spectrum as

$$S_{AM}(\omega_m) = \frac{1}{V_1^2} S_{\Delta V_1}(\omega_m) \tag{10}$$

and

$$S_{AM}(\omega_m) = \frac{\eta^2}{V_1^2} \frac{S_{\Delta V_0}(\omega_m)}{1 + (\omega_m/\omega_1)^2} + \left(\frac{1}{V_1 G_{RF}}\right)^2 \frac{S_{i_2}(\omega_m)}{1 + (\omega_m/\omega_1)^2} \tag{11}$$

where the terms used in this expression are summarized following (13). The coupling term in (8) involving ΔV_1 in the phase constraint can now be treated as a source term so that the individual spectral components of the phase can simply be added to get the phase spectrum.

At this point it is convenient to observe that the spectrum of the phase ϕ is not the quantity actually measured experimentally. Normally, the spectrum of the derivative ϕ' is measured. This is the FM-noise spectrum. The phase constraint then readily gives the FM spectrum as

$$S_{FM}(\omega_m) = S_{\phi'}(\omega_m) \tag{12}$$

and

$$S_{FM}(\omega_m) = (\eta - \xi)^2 \left(\frac{B_{RF}}{q_{RF}}\right)^2 \left[\frac{1 + \left(\frac{\xi}{\eta - \xi}\frac{\omega_m}{\omega_1}\right)^2}{1 + (\omega_m/\omega_1)^2}\right] S_{\Delta V_0}(\omega_m) + \frac{S_{i_1}(\omega_m)}{q_{RF}^2} + \frac{(\omega_0 \tau_{RF}/q_{RF})^2}{1 + (\omega_m/\omega_1)^2} S_{i_2}(\omega_m) \tag{13}$$

where the following terms have been used:

$\eta = \partial_{V_0} G_d / \partial_{V_1} G_d$	RF-to-video conductance modulation sensitivity;
$\xi = \partial_{V_0} B / \partial_{V_1} B$	RF-to-video susceptance modulation sensitivity;
$\tau_{RF} = (1/\omega_0)\partial_{V_1} B / \partial_{V_1} G_d$	RF relaxation time;
$G_{RF} = V_1 \partial_{V_1} G_d$	RF incremental conductance;
$B_{RF} = V_1 \partial_{V_1} B$	RF incremental susceptance;
$q_{RF} = V_1 [C + 1/(\omega_0^2 L)]$	stored charge per RF cycle;
$\omega_1 = V_1 G_{RF} / q_{RF}$	noise rolloff frequency;
ω_m	the frequency deviation from ω_0.

Commonly, the FM spectrum is rewritten in such a manner as to introduce a Q factor (e.g., Kurokawa [5] and Sweet [14]). Since Q factors are frequently measured parameters, those formulations are convenient. However, measured Q factors are frequently ambiguous and subject to interpretation. Moreover, the suggested parameter variations are sometimes incorrect (e.g., variation of the RF component of FM noise with load conductance). Thus the FM spectrum in (13) has been expressed in terms of the more basic quantity, stored charge.

The AM spectrum is seen to have two components: one due to video noise and the other due to the in-phase quadrature RF-noise component. The FM spectrum depends on the video noise and both quadrature RF-noise components. By proper choice of the conductance- and susceptance-modulation sensitivities the video components of AM and FM noise can be minimized. The AM video component is minimized by minimizing the conductance-modulation sensitivity η while the FM video component for constant susceptance-modulation sensitivity ξ is minimized by minimizing the expression $(\eta - \xi)^2\{1 + [\xi/(n - \xi)]^2(\omega_m/\omega_1)^2\}$ from (13). In the limit of the frequency difference ω_m from the carrier approaching zero, the FM-noise video component is minimized by approximately equating η, the conductance-modulation sensitivity, to ξ, the susceptance-modulation sensitivity. Approximately equating η to ξ is emphasized because when $\eta = \xi$ the preceding expression becomes $\xi^2(\omega_m^2/\omega_1^2)$ which is not the minimum. Both the AM and FM video components are small when both η and ξ are very small. In terms of device admittance, η and ξ are small when admittance variations with RF voltage are large and when admittance variations with bias voltage are small.

The FM video component is also dependent on the stored charge in the RF circuit. Increasing that stored charge decreases the FM noise.

Since the video component of noise typically dominates TE oscillator noise below 100 kHz from the carrier, the preceding variations of AM and FM spectra are most important for applications in this frequency range. Device fabrication for lowest noise performance should not only be directed toward improved processing techniques which reduce crystal imperfections and surface traps which lead to high $1/f$ noise, but also toward tailoring the device admittance curves so that at the desired operating point for a particular application the admittance RF-voltage variation is large and bias-voltage variation is small.

The variations of RF components of AM and FM noise describe the dependence of TE oscillator noise in the limit of negligible video or $1/f$ noise. This would be the case if fabrication techniques were refined sufficiently to virtually eliminate $1/f$ noise or if the device RF noise rose such as happens when a device is on the verge of changing modes. Further, the noise properties of devices such as IMPATT's whose oscillator noise is mainly due to RF-noise sources would be described by the RF components.

The variations of the RF components are simpler than the variations of the video-noise components and the AM- and FM-RF-noise components are uncoupled. As with the video component of FM noise, the RF components of FM noise decrease with increasing RF circuit stored charge. For the quadrature-phase RF component, stored charge is the only parameter dependence. The in-phase RF component of FM is also proportional to the RF relaxation time τ_{RF}. This means that the in-phase component is reduced by decreasing the RF-voltage slope of susceptance and increasing the RF-voltage slope of conductance. However, for $\omega_0 \tau_{RF} \ll 1$ the quadrature-phase component of FM dominates and further optimization of the slopes has little effect on the FM spectrum.

The RF component of AM noise is proportional to the reciprocal square of the conductance RF-voltage slope, and thus increasing this slope decreases the AM spectrum due to RF noise.

Reviewing the results of Sweet, his expression for the AM spectrum is the same as (11) indicating that introducing an RF-voltage dependence in the nonlinear device does not alter its AM-noise spectrum. Also his expression for the FM spectrum reveals that introducing an RF-voltage dependence leads to a correlation between the RF component of AM and FM noise because it causes noise-quadrature components to contribute to the FM noise. Further, the video component of FM noise is altered by the RF-voltage dependence.

Conclusions

Introducing both RF- and bias-voltage dependences in the admittance of the nonlinear TE device gives the same AM-noise spectrum as the case of a device admittance dependent only on bias voltage, but leads to a correlation between the RF component of AM and FM noise. This is because both quadrature components of RF noise contribute to FM noise when the device admittance is RF and bias voltage dependent. Further, the video component of FM noise is altered by the RF-voltage dependence.

The results indicate that device fabrication for lowest noise performance should not only be directed toward improved processing techniques which reduce crystal imperfections and surface traps, both of which lead to high $1/f$ noise, but also toward tailoring the device admittance curves so that at the desired operating point for a particular application the RF-voltage variation cf the admittance is large and the bias-voltage variation is small.

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On Some Integral Relationships for Commensurate Transmission-Line Networks

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Abstract—**Several integral relationships are presented for commensurate transmission-line networks. The integrals focus on the fact that $Z(1)$ for such networks, where $Z(S)$ is the input immittance of the network, is associated with a real or redundant unit element prefacing the network. Three bandwidth restrictions are derived. Some applications of the integral relationships are presented.**

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For commensurate transmission-line networks it is convenient to use Richards [1] variable S, where

- S $\tanh\{\tau s/2\} = \sum + i\Omega$;
- τ $2l/v$, the round-trip delay for the shortest commensurate length line;
- l length of the shortest commensurate length line;
- v velocity of propagation;
- s $\sigma + i\omega$, the complex frequency variable of lumped-element networks.

Richards proved that driving-point immittances (impedances or admittances) $Z(S)$ are rational functions of S and are positive real. In this short paper we consider several integral relationships for general immittance functions $Z(S)$ expressible in the form

$$Z(S) = F(S) + M(S)$$

$$F(S) = \text{Foster preamble}$$

$$= A_1^{\infty} S + \frac{A_{-1}^{0}}{S} + \sum_{k=1}^{L} \frac{2A_{-1}^{k} S}{S^2 + \Omega_k^2}$$

$$M(S) = \frac{a_0 + a_1 S + a_2 S^2 + \cdots + a_n S^n}{b_0 + b_1 S + b_2 S^2 + \cdots + b_m S^m}, \qquad n = m \text{ or } m - 1.$$

Also, at infinity, $M(S)$ can be expanded into

$$\lim_{S \to \infty} M(S) = m_\infty + \frac{m_{-1}}{S} + \frac{m_{-2}}{S^2} + \cdots.$$

The first integral, and one of primary interest, is

$$\oint_C \frac{Z(S)}{S^2 - 1}\, dS \tag{1}$$

where C is the Bromwich [2] contour consisting of the $\sum = 0$ axis, and the infinite semicircle enclosing the RHP. By Cauchy's theorem, (1) is

$$\pi i Z(1) = \int_{i\infty}^{-i\infty} \frac{Z(i\Omega) i\, d\Omega}{-\Omega^2 - 1} + \oint_{-\pi/2}^{\pi/2} \frac{Z(S)\, dS}{S^2 - 1}.$$

Details of evaluating the RHS of the previous equation are given in the Appendix. The final result is

$$Z(1) = \frac{2}{\pi} \int_0^{\infty} \frac{R(\Omega)\, d\Omega}{\Omega^2 + 1} + A_1^{\infty} + A_{-1}^{0} + 2 \sum_{k=1}^{L} \frac{A_{-1}^{k}}{1 + \Omega_k^2} \tag{2}$$

where $R(\Omega) + iX(\Omega) = Z(i\Omega)$. On the "real-frequency axis"

$$S = i\Omega = i \tan\theta$$

where $\theta = \omega l/v$ is the electrical length. Substitution into (2) results in

$$Z(1) = \frac{2}{\pi} \int_0^{\pi/2} R(\theta)\, d\theta + A_1^{\infty} + A_{-1}^{0} + 2 \sum_{k=1}^{L} \frac{A_{-1}^{k}}{1 + \Omega_k^2}. \tag{3}$$

The integral on the RHS of (3) is the average of $R(\theta)$ over $\pi/2$ rad. Hence, transposing, (3) states

$$R_{\text{avg}} = Z(1) - 2 \sum_{k=1}^{L} \frac{A_{-1}^{k}}{1 + \Omega_k^2} - A_{-1}^{0} - A_1^{\infty}. \tag{4}$$

Thus, the average value of the real part of $Z(S)$ over $\pi/2$ rad equals $Z(1)$ less the weighted values of the residues of its Foster preamble. Equations (2)–(4) are particularly useful forms since